1 The Theorem Presented in Class

Theorem 1.

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4} \tag{1}$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{n} k(k+1)(k+2) &= \sum_{k=1}^{n}(k^3+3k^2+2k) && (2)\\
&= \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k && (3)\\
&= \frac{n^2(n+1)^2}{4} + 3\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} && (4)\\
&= \frac{n(n+1)}{4}\{n(n+1)+2(2n+1)+4\} && (5)\\
&= \frac{n(n+1)}{4}\left\{n^2+5n+6\right\} && (6)\\
&= \frac{n(n+1)(n+2)(n+3)}{4} && (7)
\end{aligned}$$

□

2 Sou's Generalization

Theorem 2.

$$\sum_{k=1}^{n} k(k+1)\cdots(k+r) = \frac{n(n+1)\cdots(n+r+1)}{r+2} \tag{8}$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{n} k(k+1)\cdots(k+r) &= \sum_{k=1}^{n} k(k+1)\cdots(k+r)\left\{\frac{(k+r+1)-(k-1)}{r+2}\right\} && (9)\\
&= \frac{1}{r+2}\sum_{k=1}^{n}\{k(k+1)\cdots(k+r+1)-(k-1)k(k+1)\cdots(k+r)\} && (10)\\
&= \frac{n(n+1)\cdots(n+r+1)}{r+2} && (11)
\end{aligned}$$

□

3 Alternate Proof of Sou's Generalization

Lemma 3 (Pascal's second identity)**.**

$$\sum_{k=0}^{n} \binom{k+m}{k} = \binom{n+m+1}{n} \tag{12}$$

Proof. We use induction on n. For the base case, $n = 0$:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{k+m}{k} &= \sum_{k=0}^{0} \binom{k+m}{k} &(13)\\
&= \binom{m}{0} &(14)\\
&= 1 &(15)\\
&= \binom{m+1}{0} &(16)\\
&= \binom{n+m+1}{n} &(17)
\end{aligned}$$

Assume for the purpose of induction that for some $n \geq 0$, $\sum_{k=0}^{n} \binom{k+m}{k} = \binom{n+m+1}{n}$. Let us show that the statement also holds for $n+1$:

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{k+m}{k} &= \sum_{k=0}^{n} \binom{k+m}{k} + \binom{n+m+1}{n+1} &(18)\\
&= \binom{n+m+1}{n} + \binom{n+m+1}{n+1} &(19)\\
&= \frac{(n+m+1)!}{n!(m+1)!} + \frac{(n+m+1)!}{(n+1)!m!} &(20)\\
&= \frac{(n+m+1)!}{n!m!(m+1)} + \frac{(n+m+1)!}{n!(n+1)m!} &(21)\\
&= \frac{(n+m+1)!}{n!m!} \left[\frac{1}{m+1} + \frac{1}{n+1} \right] &(22)\\
&= \frac{(n+m+1)!}{n!m!} \left[\frac{n+m+2}{(n+1)(m+1)} \right] &(23)\\
&= \frac{(n+m+2)!}{(n+1)!(m+1)!} &(24)\\
&= \binom{n+m+2}{n+1} &(25)
\end{aligned}$$

□

Proof of main theorem.

$$\sum_{k=1}^{n} k(k+1)\cdots(k+r) = \sum_{k=1}^{n} \frac{(k+r)!}{(k-1)!} \tag{26}$$

$$= \sum_{k=0}^{n-1} \frac{(k+r+1)!}{k!} \tag{27}$$

$$= (r+1)! \sum_{k=0}^{n-1} \frac{(k+r+1)!}{k!(r+1)!} \tag{28}$$

$$= (r+1)! \sum_{k=0}^{n-1} \binom{k+r+1}{k} \tag{29}$$

$$= (r+1)! \binom{n+r+1}{n-1} \tag{30}$$

$$= (r+1)! \frac{(n+r+1)!}{(n-1)!(r+2)!} \tag{31}$$

$$= \frac{1}{r+2} \left[\frac{(n+r+1)!}{(n-1)!} \right] \tag{32}$$

$$= \frac{n(n+1)\cdots(n+r+1)}{r+2} \tag{33}$$

□

4 The Easy Alternate Proof

Second alternate proof. We use induction.

Base case. $(n=0)$

$$\sum_{k=1}^{0} k(k+1)\cdots(k+r) = 0 \tag{34}$$

$$= \frac{0(1)\cdots(r+1)}{r+2} \tag{35}$$

Inductive step. Assume $\sum_{k=1}^{n} k(k+1)\cdots(k+r) = \frac{n(n+1)\cdots(n+r+1)}{r+2}$.

$$\sum_{k=1}^{n+1} k(k+1)\cdots(k+r) = \sum_{k=1}^{n} k(k+1)\cdots(k+r) + [(n+1)(n+2)\cdots(n+r+1)] \tag{36}$$

$$= \frac{n(n+1)\cdots(n+r+1)}{r+2} + \frac{(r+2)(n+1)(n+2)\cdots(n+r+1)}{r+2} \tag{37}$$

$$= \frac{(n+1)\cdots(n+r+1)(n+r+2)}{r+2} \tag{38}$$

□